Variations of the asset prices

Michael Schulz, Steffen Trimper, and Beatrix Schulz
Fachbereich Physik, Martin-Luther-Universität, D-06099 Halle, Germany
(Received 26 September 2000; published 17 July 2001)

The empirical established non-Gaussian behavior of asset price fluctuations is studied using an analytical approach. The analysis is based on a nonlinear Fokker-Planck equation with a self-organized feedback-coupling term, devised as a fundamental model for price dynamics. The evidence, and the analytical form of the memory term, are discussed in the context of statistical physics. It will be suggested that the memory term in leading order offers a power law dependence with an exponent θ. The stationary solution of the probability density leads asymptotically to a truncated Lévy distribution, the characteristic exponent β of which is related to the exponent θ by $\beta = 3/\theta - 1$. The empirical data can be reproduced by $\theta \simeq 5/4$.

DOI: 10.1103/PhysRevE.64.026104 PACS number(s): 05.40.Fb, 05.45.Tp, 89.90.+n

I. INTRODUCTION

In recent years, there has been increasing interest in applying concepts and methods of statistical physics to study problems of the financial market [1–3] and other complex systems from heartbeats to weather [4], to politics [5], to medical care [6], and further to ecology [7]. Similar to statistical physics the abovementioned systems, such as economic ones, consist of a large number of interacting units (‘‘agents’’). Hence experiences gained by studying complex physical systems might yield new results in economics. However, agents making financial transactions believe in ‘‘units,’’ the interaction of which is not quantified in detail. Consequently, economic systems are quite different, and much more complex. Nevertheless, the evolution of financial data should be governed by probabilistic laws well known in statistical mechanics [8]. Apparently, various financial time series undergo random processes as particles make Brownian motions. Hence one of the reasons to analyze financial systems by methods developed for physical systems is the challenge of understanding the dynamics of a strongly fluctuating system with a large number of interacting elements [9]. Moreover, simple models are discussed whose origin lies in market scenario [10], and they yield an unusual type of microdynamics of more general interest. Another approach consists of modeling the dynamics of money directly [11], where the present amount of money is a time dependent quantity, the dynamics of which is studied by an evolution equation.

The interacting elements of financial markets can be grouped into two categories: the traders, such as mutual funds, brokerage firms, banks, and individual investors; and assets, such as stocks, bonds, futures, and options. In this context, the statistical properties of the time evolution of the price play an important role in the modeling of financial markets. For example, the stochastic nature of the price of a financial asset is crucial for a rational pricing of a derivative product issued on it [3]. The initial price of a new product, a new service, or a new stock certificate is determined by certain economical guidelines. But after the introduction of the new product into the market system, the actual price starts to fluctuate due to the interaction between supply and demand. Hence the price leaves the initial value, and offers a stochastic behavior. The complete characterization of such a random process requires a knowledge of the total probability distribution density. The difficulties in quantifying the fluctuations of asset prices, originated by many interactions between the market elements considered above, are not only due to the complexity of the internal elements but also to the many intractable external factors acting on it. To this aim, models should be studied which capture the key features of the financial markets. Here we are interested in the time evolution of $S(t)$ as the price of a financial asset at time t. The most common model of price dynamics, known as the geometric Brownian motion (for details and history, compare Ref. [1]) assumes that $\ln S(t)$ is a diffusive process. However, such an approach provides only a first step toward the behavior observed in empirical data. Systematic deviations from the simple model predictions were observed [12]. A more detailed discussion can be found, for instance, in Refs. [1,3] and the literature cited there. Several alternative models beyond geometric Brownian motion were proposed such as the Lévy stable non-Gaussian model [13], Student's t distribution [14]; and a mixture of Gaussian distribution [15] and the truncated Lévy flight, either abrupt [16] or smooth [17] or scale invariant [18]; for other models, see Ref. [3]. There seems to be well established evidence that the distribution function in mind offers a non-Gaussian behavior, the center of which can be adjusted by Lévy distribution. Consequently, the price volatility reveals a nondiffusive behavior at least on a small time scale. Although the distribution is long tailed, it reveals finite moments of any order [19].

Whereas most previous approaches were based on an assumption of the price probability distribution $P(t)$, and were quantified by numerical investigations, the aim of this paper is to propose a dynamical equation for $P(t)$. Our ambition follows the line given by Black and Scholes [20], which was successful in deriving a partial differential equation for a geometrical Brownian process. However, this equation does not take memory effects into account. On the other hand, such a feedback mechanism was considered in econophysics by so called ARCH [21] and GARCH processes [22] in a first approximation. Because price fluctuations at time t are likewise determined by changing prices at previous time $t' < t$, the evolution equation for the probability distribution $P(t)$ should include a memory term as a new ingredient. Moreover, we have supposed that this feedback is determined in a self-organized manner by the price distribution itself. As demonstrated in Ref. [23], such a feedback coupling can give rise to anomalous diffusion or localization. Note that a relation between economic fluctuations and anomalous diffusion was even discussed [24]. The basic equation we have proposed is derived in the context of statistical physics by applying a projection formalism due to Mori [25]. As the result a nonlinear evolution equation of Fokker-Planck type with an additional memory kernel is found, the form and relevance of which are studied in this paper.

II. MODEL

A. Return

In this section the central quantity of the model is introduced, and its relation to a linear Fokker-Planck equation is discussed. As stressed above, $S(t)$, the price of a financial asset at time t, is the fundamental function for subsequent analysis. Because, after an initial time interval, the price becomes a fluctuating quantity, it is reasonable to investigate the probability distribution function of an appropriate stochastic variable. One choice is to analyze the return (see Ref. [1]),

$$l(t)=\ln\left[\frac{S(t+\Delta t)}{S(t)}\right], \tag{1}$$

where Δt is a well defined short time interval. Assuming that trading is continuous and the limit $\Delta t\to 0$ exists, it seems to be reasonable to use the velocity $v(t)=l(t)/\Delta t\cong\partial \ln S(t)/\partial t$ as a quantity to characterize asset price fluctuations. The time evolution of $v(t)$ can be described when the probability density function $P(v,t)$ is available. Here $P(v,t)dv$ is the probability of finding a velocity between v and $v+dv$. Immediately after the introduction of the new product or the new stock, the velocity v apparently shows a systematic change following global trends. Furthermore, one expects the appearance of Gaussian-like fluctuations around this deterministic behavior. Thus $P(v,t)$ should follow a Fokker–Planck equation:

$$\frac{\partial P(v,t)}{\partial t}=D\frac{\partial^2 P(v,t)}{\partial v^2}-\frac{\partial}{\partial v}[F(v)P(v,t)]. \tag{2}$$

The diffusion coefficient D determines the random motion of the velocity v, whereas the drift force $F(v)$ defines the general trend of the price evolution. Here it is supposed that the trend is constant, at least approximately for a sufficient long time interval, i.e., the drift force is assumed to be fixed $F(v)=F_0$. Equation (2) will be problematic considering its asymptotic behavior in the long time limit, where Eq. (2) leads to a Gaussian behavior corresponding to $P(v,t)\sim\exp(-[v+F_0t]^2/4Dt)$. Such a behavior is not observed by empirical data [12,13]. Instead, large price fluctuations occur much more frequently than predicted by the Gaussian law. We think that these deviations originated from the strong coupling among all prices, and the interaction with other economical degrees of freedom which cannot be specified in detail. This fact is a characteristic feature of a variety of different complex systems.

B. Nonlinear Fokker-Planck equation

In this subsection a generalized equation is proposed where couplings, such as those mentioned above are enclosed. The situation for the asset price fluctuations is comparable with other problems in statistical physics of complex systems. In case information for all degrees of freedom is available, one could formally derive an evolution equation for the total probability density function. For instance, the Liouville equation is such an evolution equation describing thermodynamic systems. A general property of those equations is the local character with respect to all degrees of freedom and time. Unfortunately, neither in physics nor in financial systems can complete knowledge of all constituent elements be quantified. Even the set of all prices comprises only a sufficiently small subset of all degrees of freedom, relevant for the dynamics of an econosystem. According to a standard method of statistical mechanics, let us eliminate all unknown or irrelevant degrees of freedom by a suitable projection operator formalism [25]. Note that the procedure is universal, and does not depend on special realizations of the underlying complex systems such as many-particle systems, financial markets, social populations, etc. The application of an appropriately chosen projection operator [25] on the total probability distribution function leads to the so called Nakajima–Zwanzig equation [26], which describes the evolution of a reduced distribution function in terms of the remaining relevant variables. It should be stressed that the resulting equation is rigorous now as before. In our model the variable v is assumed to be the relevant one. Consequently, the equation for $P(v,t)$ reads

$$\partial_t P(v,t)=\hat{M}(v,t)P(v,t)-\int_0^t dt'\int_{-\infty}^{\infty}dv'\hat{K}(v-v',t-t')P(v',t'). \tag{3}$$

Both the operator $\hat{M}(v,t)$ and the memory kernel $\hat{K}(v,t)$ are well defined formal expressions which determine the dynamics of the probability density function. The procedure yields the required memory effect in a natural manner. In this sense Eq. (3) can be regarded as a more general approach as the above mentioned ARCH and GARCH processes. To proceed further, one needs a suitable representation of $\hat{M}$ and $\hat{K}$, respectively. If memory effects are excluded completely, all other degrees of freedom projected out are apparently independent external stochastic variables. Thus the first term of Eq. (3) can be taken as the Fokker-Planck operator in according to Eq. (2),

$$\hat{M}(v,t)P(v,t)=D\frac{\partial^2 P(v,t)}{\partial v^2}-\frac{\partial}{\partial v}[F(v)P(v,t)], \tag{4}$$

with the diffusion coefficient D and the drift force $F(v,t)$. The memory term takes into account the history of the price evolution. Formally, the kernel can be rewritten as

$$\int_0^t dt' \int_{-\infty}^{\infty} dv' \hat{K}(v-v',t-t') P(v',t') = \int_0^t dt' \int_{-\infty}^{\infty} dv' K(v-v',t-t') \frac{\partial P(v',t')}{\partial t'}. \quad (5)$$

One can easily confirm that Eq. (5) guarantees the conservation of the probability $\int dv P(v,t)=1$ at all times t. For this purpose Eq. (3) is integrated over all velocities v, and Eqs. (4) and (5) are taken into account. Introducing $P_0(t) = \int_{-\infty}^{\infty} dv P(v,t)$,

$$\partial_t P_0(t) = -\int_0^t dt' \int_{-\infty}^{\infty} dv K(v,t-t') \frac{\partial P_0(t')}{\partial t'}. \quad (6)$$

The Laplace transformation of Eq. (6), with respect to the time, leads to

$$zP_0(z) - P_0(t=0) = -\int_{-\infty}^{\infty} dv K(v,z)[zP_0(z) - P_0(t=0)], \quad (7)$$

which has the two solutions: $zP_0(z)=P_0(t=0)$ and $\int_{-\infty}^{\infty} dv K(v,z) = -1$. Because the second one is not a reliable solution, only the first solution is relevant for further considerations. But this relation corresponds to the above mentioned conservation law $\int dv P(v,t)=1$. Together, Eqs. (3)–(5), with the specified memory kernel (see Sec. III), are the basic equations for the subsequent analysis.

III. RESULTS

A. Memory kernel

To study the influence of the memory effects in detail, we have to specify the memory kernel $K(v,t)$. It is not yet entirely clear how one should approach the problem. But one can gain insight by learning from such completely different complex systems as undercooled liquids, where the concept considered in this paper was applied successfully in explaining properties of dense systems [27,28]. An idea similar to arguments used in mode coupling theory was also recently adopted to study anomalous diffusion [23,29,30]. The main assumption for our model is that feedback originates from a coupling among prices, and consequently the kernel can be considered as $K(v,t)=K[P(v,t)]$. This relation implies the reasonable claim that the characteristic time scale of the probability function also the time scale of the memory essentially determines, in other words, if the prices fluctuate during a certain time interval, then the memory term should relax within the same order of magnitude. As a further consequence of the assumption, the structure of the kernel leads to a nonlinear autonomous equation which is a feature of systems incorporating scaling behavior and self-organization. Now let us expand the memory kernel $K[P(v,t)]$ in terms of the probability distribution $P(v,t)$. The general form of such an expansion can be written in the form

$$K(v,t) = P(v,t)^{\theta} \sum_{n=0}^{\infty} \lambda_n P(v,t)^n, \quad (8)$$

where θ is an exponent specified in the subsequent discussion. Note that in the case of undercooled liquids the projection procedure suggests an exponent $\theta=2$ [27,28], which is not maintained necessarily when the conceptual framework is applied to financial systems. Finally, Eqs. (3)–(5), combined with the leading term of Eq. (8) yield a generalized nonlinear Fokker-Planck equation

$$\frac{\partial P(v,t)}{\partial t} = D \frac{\partial^2 P(v,t)}{\partial v^2} - \frac{\partial}{\partial v}[F_0 P(v,t)] - \lambda \int_0^t dt' \int_{-\infty}^{\infty} dv' P^{\theta}(v-v',t-t') \frac{\partial P(v',t')}{\partial t'} \quad (9)$$

[with $\lambda=\lambda_0$ and $F(v,t)=F_0$]. Equation (9) is similar to that given by Black and Scholes [20]; however our basic equation includes an additional nonlinear memory term describing the required feedback mechanism. A conventional scaling transformation for P, v, and t offers that the memory term is relevant for $\theta<3$. A more detailed analysis of Eq. (9) with a d-dimensional vector $\mathbf{v}$ instead of the component v suggests a critical dimension $d_c = 2/(\theta-1)$. Below this critical dimension, the occurrence of anomalous behavior is expected. At first view it seems to be an obvious assumption that memory kernel [Eq. (8)] behaves regularly, i.e., θ is a non-negative integer exponent. Due to the projection formalism [25], $\theta=0$ and 1 are excluded. Consequently, only $\theta=2$ remains. This case, as already pointed out, is well known in studying undercooled liquids [27,28] and very recently in analyzing anomalous diffusion in a glasslike environment [23]. Therefore, it is useful to summarize some results of the previous approaches [23,29,30], which are relevant in proceeding further in the present paper. In particular, the case $\theta=2$, leading to the critical dimension $d_c=2$, can be studied using a perturbative renormalization group method [23]. The analysis offers the existence of several anomalous diffusion regimes, induced by the strength of the coupling constant λ and the realization of the drift force. For example, for a vanishing drift term and $\lambda<0$ a superdiffusive behavior was found to be manifested by the behavior of the mean square displacement $\overline{\mathbf{v}^2} \sim t^{2/z}$ with a dynamical exponent $z<2$, whereas a positive coupling $\lambda>0$ gave rise to localization evident by the relation $\overline{\mathbf{v}^2} \sim$ const. Whenever $\theta=2$, Eq. (9) can likewise be analyzed numerically by mapping the problem on a simple lattice model with hopping processes under the inclusion of a feedback mechanism [29,30]. In case of a negative feedback coupling $\lambda<0$, the numerical simulations confirmed the superdiffusive behavior with $\overline{v^2} \sim t^{1.37}$ for $d=1$ [29], which is in reasonable agreement with the one loop

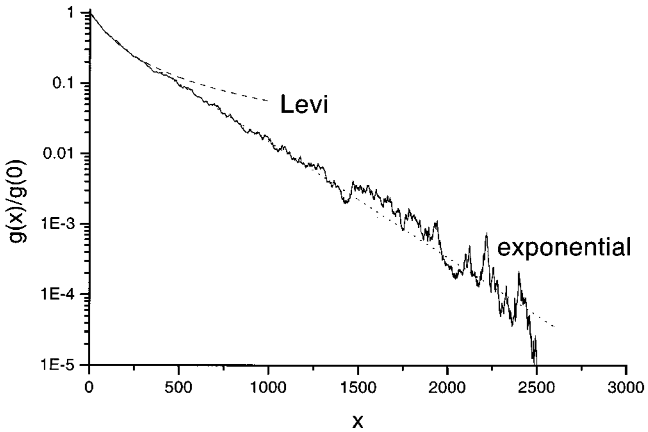

FIG. 1. Numerical simulation of the stationary probability distribution $g(x)$ for a random process corresponding to Eq. (9) with $F_0=0$, $\theta=2$, and $\lambda>0$. For small x, denoted by "Levi," a Lévy distribution with the exponent $\beta\simeq 0.34$ is observed, whereas for large x, denoted by "exponential," an exponential distribution is realized.

renormalization group approach $2/z=3/2$ [23]. In two dimensions both renormalization group and numerical simulations [30] led to a diffusive behavior with a logarithmic correction $\overline{\mathbf{v}^2}\sim t\ln t$. For a positive coupling $\lambda>0$, localization was observed to be renewed. The occurrence of localization strongly indicates the convergence of the probability function $P(v,t)$ to a stable function $g(v)$ for sufficient long times $P(v,t\to\infty)=g(v)$, which are discussed now. Using the previous results [29] for $\theta=2$, one obtains an exponential decay for large v and a Lévy distribution

$$g(v)\sim[1+(v/v_0)^{\beta+1}]^{-1} \tag{10}$$

for small v. The memory exponent $\theta=2$ leads to a Lévy exponent $\beta\approx 0.34$. In Fig. 1 the numerically simulated stationary probability distribution for a random process corresponding to Eq. (9) is depicted, where $F_0=0$, $\theta=2$, and $\lambda>0$ are taken into account; for details of the simulations, see Ref. [29]. The appearance of such a stationary solution $g(v)$ reflects even the situation expected for the evolving asset price system [12]. Using the return $l=v\Delta t$ (for small Δt, e.g., $\Delta t=1$ min) as a measure for the velocity v, Mantegna and Stanley found [12], based on numerical study, a Lévy-like regime described by $g(l)\sim[1+(l/l_0)^{\beta+1}]^{-1}$ with the exponent $\beta\approx 1.4$ for small l, whereas an exponential decay $g(l)\sim\exp(-\gamma l)$ occurs for large l. Obviously, both approaches [12], and those presented for $\theta=2$, indicate a universal behavior for $g(v)$ which cannot be explained using a simple Gaussian model. But apparently the exponent $\beta\approx 1.4$ is much larger than the exponent $\beta\approx 0.34$ obtained for $\theta=2$. However, as already stressed above, it cannot be excluded that financial markets will be controlled by a noninteger exponent θ in the memory kernel; this is discussed in Sec. III B.

B. Asymptotic solution

In this section a relation between the memory exponent θ and the Lévy exponent β is derived. To this aim the asymptotic distribution function $g(v)$ will be determined, the existence of which was already argued above. In a first step let us eliminate the drift term from Eq. (9). This term represents the mean change of the price fluctuations, i.e., the trend of the time evolution of prices. As mentioned above, the drift force is a constant quantity F_0. Making a simple shift $v\to x-F_0 t$ the evolution equation for the price fluctuations reads

$$\frac{\partial P(x,t)}{\partial t}=D\frac{\partial^2 P(x,t)}{\partial x^2}-\lambda\int_0^t dt' \times\int_{-\infty}^{\infty}dx' P^{\theta}(x-x',t-t')\frac{\partial P(x',t')}{\partial t'}. \tag{11}$$

The probability distribution function $P(x,t)$ converges to a stationary distribution $g(x)$. Because we are interested in the asymptotic limit for long times, let us make the ansatz

$$P(x,t)=g(x)+\kappa(x,t), \tag{12}$$

with $\kappa(x,t)\to 0$ for $t\to\infty$. The Laplace transformation of Eq. (12) with respect to time leads to $P(x,z)=g(x)/z+\kappa(x,z)$, where $\kappa(x,z)$ is regular for $z\to 0$. Thus the singular part of Eq. (11) for $z\to 0$ defines an equation for the stationary solution.

$$D\frac{\partial^2 g(x)}{\partial x^2}=\lambda\int_{-\infty}^{\infty}dx' g^{\theta}(x-x')g(x')-\lambda g^{\theta}(x). \tag{13}$$

Note that Eq. (13) follows from Eq. (11) after a Laplace transformation using the initial condition $P(x,0)=\delta(x)$. The Fourier transformation of Eq. (13) with respect to the normalized price fluctuations x leads to

$$Dk^2 g(k)=-\lambda g^{(\theta)}(k)g(k)+\lambda g^{(\theta)}(k), \tag{14}$$

where $g^{(\theta)}(k)$ is the Fourier transform of $g^{\theta}(x)$. Hence the formal solution for $g(k)$ is given by

$$g(k)=\frac{\lambda g^{(\theta)}(k)}{Dk^2+\lambda g^{(\theta)}(k)}. \tag{15}$$

Especially, for $k\to 0$, it follows that

$$g(k)=\left[1+\frac{Dk^2}{\lambda g^{(\theta)}(0)}\right]^{-1}. \tag{16}$$

This solution corresponds to the asymptotic behavior for large $x\to\infty$

$$g(x)\sim\left(\frac{\lambda g^{(\theta)}(0)}{4D}\right)^{1/2}\exp\left\{-\left(\frac{\lambda g^{(\theta)}(0)}{D}\right)^{1/2}|x|\right\}, \tag{17}$$

i.e., the exponential behavior for large x is guaranteed for all exponents θ provided $g^{(\theta)}(0)$ is finite. The expansion $g^{(\theta)}(k)=g^{(\theta)}(0)+c_1k^2+\cdots$ cannot be used for the discussion of large k and small x, respectively. Here a power law $g(x)\sim x^{-1-\beta}$, i.e., $g(k)\sim k^{\beta}$ and $g^{(\theta)}(k)\sim k^{\theta(\beta+1)-1}$, is assumed. A simple power counting leads to the relation $\beta=3/\theta-1$. This relation is again consistent with the requirement $\theta<3$. Because of the fact that real price variations are characterized by a Lévy exponent $\beta\simeq 1.4$ [12], a memory exponent $\theta\simeq 5/4$ is concluded.

These results show that financial processes can be described by evolution equations containing memory terms. These terms may have a more complicated structure [compare Eqs. (8) and (9)], but the asymptotic behavior is well reflected by the leading term in Eq. (8).

IV. CONCLUSIONS

In this paper an attempt at an analytical approach for the description of market dynamics is proposed. In particular, a dynamical model for price distribution is considered. As the main ingredient, a nonlinear memory term is included, responsible for the feedback coupling of the momentary price distribution function to its change in the past. Such a feedback coupling is a feature of a dynamical complex system. A systematic approach including memory effects is still open. However, our paper reveals that memory effects can lead to dramatic changes in the behavior as well as in the long and the short time limits. The empirically well established observation of a non-Gaussian distribution is confirmed in a striking manner. The numerical finding [12] is also in accordance with analytical results based on a generalized Fokker-Planck equation with a self-consistent memory term. Universal properties of the price fluctuations are achieved if the memory kernel offers a power law behavior with respect to the probability density. Apparently a rigorous model should include not only the memory kernel in leading order presented by Eq. (9), but also higher order terms according to the expansion in Eq. (8). The main result of our paper is to show that, if memory effects are relevant for market scenarios, the asymptotic behavior of small and large price fluctuations can be explained by a nonlinear Fokker-Planck equation. Up to now it is not entirely clear which complementary principle in economics determines the memory exponent θ. However, our analysis suggests an exponent $\theta\simeq 5/4$.

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